Polyakov Loops, Z(N) Symmetry, and Sine-Law Scaling

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We construct an effective action for Polyakov loops using the eigenvalues of the Polyakov loops as the fundamental variables. We assume $Z(N)$ symmetry in the confined phase, a finite difference in energy densities between the confined and deconfined phases as $T \to 0$, and a smooth connection to perturbation theory for large T. The low-temperature phase consists of $N-1$ fields fluctuating around an explicitly $Z(N)$ symmetric background. In the low-temperature phase, the effective action yields non-zero string tensions for all representations with non-trivial N-ality. Mixing occurs naturally between representations of the same N-ality. Sine-law scaling emerges as a special case, associated with nearest-neighbor interactions between Polyakov loop eigenvalues.

Issues to be Addressed

- Construction of general effective action with minimal assumptions
- Comprehensive model for confined phase, deconfined phase, and phase transition
- Derivation of confinement from Z(N) symmetry for all representations of non-trivial N-ality
- Special nature of Sine-law scaling

Basics of Finite Temperature Physics

Low Temperature Behavior

$$\langle Tr_F P^n(x) Tr_F P^{+n}(y)\rangle \propto \exp\left[-\frac{\sigma_n}{T}|x-y|\right]$$

High Temperature Behavior

$$\langle Tr_F P\rangle = mz^p \qquad\qquad z \in Z(N)$$

Possible Scaling Behaviors

Casimir Scaling

$$\sigma_R = \frac{C_R}{C_F}\sigma_F$$

where C_R is the quadratic Casimir invariant for the representation R. Appears to work well for $SU(3)_4$ (*Deldar* 2000; *Bali* 2000).

***Z*(*N*) Scaling**

$$\sigma_R = \frac{k(N-k)}{N-1}\sigma_F$$

where k is the smaller of N-alities of the representations R and $\overline{R}$: $k = \min(k_R, k_{\overline{R}})$. $Z(N)$ scaling should be obtained from Casimir scaling as $r \to \infty$ if there is mixing between representations of the same N-ality. For the first $\left[\frac{N}{2}\right]$ antisymmetric representations made by stacking boxes in Young tableux, Casimir and $Z(N)$ scaling are identical.

Sine-law scaling

$$\sigma_R = \frac{\sin\left(\frac{\pi k}{N}\right)}{\sin\left(\frac{\pi}{N}\right)}\sigma_F$$

was obtained by *Douglas and Shenker* (1995) for softly broken $N = 2$ super Yang-Mills theories and by *Hanany*, *Strassler*, *and Zaffaroni* (1998) for MQCD.

SImulation Results

	k=2	k=3	k=4	Ref
SU(4) sine law	1.414			
SU(4) Z(N)	1.333			
$SU(4)_4$	1.370(20)			LTW
$SU(4)_4$	1.403(15)			DPRV
$SU(4)_3$	1.3548(64)			LT
SU(6) sine law	1.732	2		
SU(6) Z(N)	1.6	1.8		
$SU(6)_4$	1.675(31)	1.886(61)		LTW
$SU(6)_4$	1.72(3)	1.99(7)		DPRV
$SU(6)_3$	1.6160(86)	1.808(25)		LT
SU(8) sine law	1.848	2.414	2.613	
SU(8) Z(N)	1.714	2.143	2.286	
$SU(8)_4$	1.779(51)	2.38(1)	2.69(17)	LTW

LTW= *Lucini, Teper, and Wenger* (2004)
LT = *Lucini and Teper* (2001,2002)
DPRV = *Del Debbio, Pangopoulos, Rossi, and Vicani* (2002)

Multiple Order Parameters

We know there are an infinite number of possible Polyakov loop order parameters. Every irreducible representation R has an associated N-ality k_R such that

$$Tr_R P \to z^{k_R} Tr_R P$$

under a global $Z(N)$ transformation, with $k_R \in \{0,..,N-1\}$. Any representation with $k_R \neq 0$ gives an operator $Tr_R P$ which is an order parameter for the spontaneous breaking of the global $Z(N)$ symmetry. Alternatively, we may consider the operators $Tr_F P^n$ which transform as

$$Tr_F P^n \to z^n Tr_F P^n.$$

These operators transform non-trivially under $Z(N)$ provided n is not a multiple of N.

In a gauge where A_0 is time independent and diagonal, in which case we may write for the fundamental representation

$$P_{jk} = \exp(i\theta_j)\delta_{jk}$$

where we shall refer to the N numbers θ_j as the eigenvalues. They are not independent because $\det(L) = 1$ implies

$$\sum_j \theta_j = 0 \bmod 2\pi$$

The information in the different representations is redundant. All the information is contained in the $N-1$ independent eigenvalues of P.

In constructing an effective action, we cannot use P, regarded as an $SU(N)$ matrix, as the fundamental field for building an action without introducing spatial gauge fields, lest we introduce spurious Goldstone bosons in the deconfined phase associated with the off-diagonal components. We cannot use $Tr_F P$ alone, because it does not specify the other Polyakov loops for $N > 3$. The cases of $SU(2)$ and $SU(3)$ are special because only in those two cases does $Tr_F P$ specify all the other fields. Beginning with $SU(4)$, there are sets of eigenvalues for which

$$Tr_F P = 0 \text{ and } Tr_F P^2 \neq 0$$

(*Meisinger, Miller, and Ogilvie* (2002). From the characteristic polynomial, one may show that the eigenvalues of a special unitary matrix are determined by the set $\{Tr_F P^k\}$ with $k = 1..N-1$, and of course *vice versa*.

P in the confined phase

In both the confined and deconfined phase, we would like to use our effective theory by starting from a classical field configuration which has the symmetries of the phase. In the confined phase, $Z(N)$ symmetry requires that $Tr_F P^k = 0$ for all k not divisible by N. Enforcing this requirement for $k = 1$ to $N - 1$ leads to a unique set of eigenvalues via the characteristic equation $z^N + (-1)^N = 0$, but it is instructive to derive the set another way. For temperatures T below the deconfinement transition T_d, center symmetry is unbroken. Unbroken center symmetry implies that zP is equivalent to P after an $SU(N)$ transformation:

$$zP_0 = gP_0 g^+.$$

This condition in turn implies $Tr_R P = 0$ for all representations R with non-zero N-ality, which means that all representations with non-zero N-ality are confined. The most general form for P_0 may be given as hdh^+, where $h \in SU(N)$, and d is the diagonal element of $SU(N)$ of the form

$$d = w\, diag[z, z^2, .., z^N = 1]$$

with w is a phase given by $w = \exp[-(N+1)\pi i/N]$. Strictly speaking, w is necessary only for N even, but it is convenient to use it consistently. We will henceforth identify P_0 with d. Another useful representation is $(P_0)_{jk} = \delta_{jk} \exp\left[i\theta_j^0\right]$ where

$$\theta_j^0 == \frac{\pi}{N}(2j - N - 1).$$

The $Z(N)$-symmetric arrangement of eigenvalues is uniform spacing around the unit circle.

Fluctuations around the $Z(N)$-symmetric vacuum

Let us now assume that P_0 is the global minimum of the potential V associated with the effective action for temperatures less than the deconfining temperature T_d. Because the gauge fields transform as the adjoint representation, V is a class function depending only on representations of zero N-ality. It is thus a function only of the differences in eigenvalues $\theta_j - \theta_k$, with complete permutation symmetry as well. In the low-temperature, confining phase, we consider small flucuations about P_0, defining $\theta_j = \theta_j^0 + \delta\theta_j$. For small fluctuations

$$Tr_F P^k = \sum_{n=1}^{N} w^k z^{kn} e^{ik\delta\theta_n} \simeq \sum_{n=1}^{N} w^k z^{kn} ik\delta\theta_n$$

if k is not divisible by N, and $Tr_F P^k$ takes the form of a discrete Fourier transform in eigenvalue space. We define the Fourier transform of the fields ϕ_n as

$$\phi_k = \sum_{n=1}^{N} z^{kn}\delta\theta_n$$

so $Tr_F P^k = ikw^k\phi_k$. We will show below that the fields ϕ_k are the normal modes of the $Z(N)$-symmetric phase in a quadratic approximation. The mode $\phi_N \equiv \phi_0$ is identically zero for $SU(N)$, and from the reality of θ, we have $\phi_{N-n} = (\phi_n)^*$. In the case where k is divisible by N, $Tr_F P^k$ has a leading constant behavior of Nw^k, and the term linear in ϕ vanishes.

Note for future use that operators with the same N-ality, such as $Tr_F P^k$ and $(Tr_F P)^k$ are very different operators, even at lowest order, being proportional to ϕ_k and $(\phi_1)^k$. Group characters $\chi_R(P)$ will be represented as sums of terms with the same N-ality, but with different mode content. For example, in $SU(4)$, the **10** and **6** representations are given by

$$\chi_{S,A} = \frac{1}{2}\left[(Tr_F P)^2 \pm Tr_F P^2\right] \simeq \frac{1}{2}[\pm 2\phi_2 + i\phi_1^2].$$

In general, an operator of a given N-ality may produce several different excitations, and only the lightest states will dominate at large distances. Characters are complicated functions of the underlying modes.

The transition from Casimir scaling to scaling based on N-ality is the standard field-theoretic mechanism of mixing and decay.

S_{eff} at high temperatures

The form of the effective action at high temperatures can be written as

$$S_{eff} = \beta \int d^3x \left[T^2 Tr_F (\nabla\theta)^2 + V_{1L}(\theta) \right]$$

Bhattacharya, Gocksch, Korthals Altes, and Pisarski (1992). The kinetic term is obtained from the underlying gauge action via $\frac{1}{2} Tr_F F_{\mu\nu}^2 \to \frac{1}{2} 2Tr_F (\nabla A_0)^2 \to T^2 Tr_F (\nabla\theta)^2$. The potential $V_{1L}(\theta)$ is obtained from one-loop perturbation theory. For our purposes, it is conveniently expressed as

$$\begin{aligned} V_{1L}(\theta) &= -\sum_{n=1}^{\infty} \frac{2}{\pi^2} \frac{T^4}{n^4} [|TrP^n|^2 - 1] \\ &= -\sum_{n=1}^{\infty} \frac{2}{\pi^2} \frac{T^4}{n^4} \left[N - 1 + \sum_{j \neq k} \cos(n(\theta_j - \theta_k)) \right] \end{aligned}$$

(*Gross, Pisarski, and Yaffe* 1980; *Weiss* 1981,1982). This series can be summed to a closed form in terms of the 4th Bernoulli polynomial. The complete one-loop expression has been obtained recently by *Diakonov and Oswald* (2003; 2004); the complete kinetic term has a θ-dependent factor in front of the derivatives.

There are N equivalent solutions of the form

$$\theta_j^{(p)} = \frac{2\pi p}{N}$$

related by $Z(N)$ symmetry breaking. All of these solutions break $Z(N)$ symmetry, with $Tr_F P = N \exp(2\pi i p / N)$. For these values of θ, we recover the standard black-body result for the free energy.

The form of the effective action is fixed at high temperature by perturbation theory.

General Form of S_{eff}

We will assume that a sufficiently general form of the action at all temperatures has the form

$$S_{eff} = \beta \int d^3x \left[\kappa T^2 Tr_F (\nabla\theta)^2 + V(\theta) \right]$$

where κ is a temperature dependent correction to the kinetic term, and V is a function only of the adjoint eigenvalues $\theta_j - \theta_k$.

Form of the potential term at low temperatures

We assume that there is a finite free energy density difference associated with different values of P as $T \to 0$. Because the eigenvalues are dimensionless, this requires terms in the potential with coefficients proportional to $(mass)^4$ as $T \to 0$.

We can expand the potential to quadratic order around P_0

$$V(\theta) \simeq V(\theta^0) + \sum_{j,k} \frac{1}{2} \left[\frac{\partial^2 V}{\partial\theta_j \partial\theta_k} \right]_{\theta^0} \delta\theta_j \delta\theta_k$$

where the coefficient in the expansion depends only on $|j-k|$. The large-N limit can be used to justify the assumption that fluctuations are small. The quadratic piece is thus diagonalized by the Fourier transform, and we can write

$$V(\theta) \simeq V(\theta^0) + \sum_{n=1}^{N-1} M_n^4 \phi_n \phi_{N-n}.$$

Similarly, the kinetic term becomes

$$\kappa T^2 Tr_F (\nabla\theta)^2 = \frac{\kappa T^2}{N} \sum_{n=1}^{N-1} (\nabla\phi_n)(\nabla\phi_{N-n}).$$

Once an ordering of eigenvalues is chosen, $Z(N)$ symmetry is expressed as a discrete translation symmetry in eigenvalue space. If we write the higher-order parts of S_{eff} in terms of the Fourier modes ϕ_n, each interaction will respect global conservation of N-ality. For example, in $SU(4)$, an interaction of the form $\phi_1^2\phi_2$ is allowed, but not $\phi_1^2\phi_2^2$.

Correlation Functions below T_d

If the interactions are sufficiently weak, we can calculate the behavior of Polyakov loop two-point functions from the quadratic part of S_{eff}. We have for large distances

$$\langle Tr_F P^n(x) Tr_F P^{+n}(y)\rangle \propto \langle \phi_n(x)\phi_n^*(y)\rangle \propto \exp\left[-\frac{\sigma_n}{T}|x-y|\right]$$

where $\sigma_n(T) = \sqrt{NM_n^4(T)/\kappa(T)}$ is identified as the string tension for the n'th mode at temperature T. Of course, $\phi_{N-n} = (\phi_n)^*$ implies $\sigma_n(T) = \sigma_{N-n}(T)$. The number of different string tensions is thus $\left[\frac{N}{2}\right]$, the greatest integer less than or equal to $N/2$. The zero-temperature string tension is given by

$$\sigma_n^2(0) = NM_n^4(0)/\kappa(0)$$

It is easy to see that the $\left[\frac{N}{2}\right]$ string tensions $\sigma_1(0), \sigma_2(0), .., \sigma_{[N/2]}(0)$ are all set independently within the class of effective models. A **minimal** model for the confined phase exhibiting this quadratic behavior is

$$V = \sum_{k=1}^{[N/2]} \frac{M_k^4}{k^2} Tr_F P^k Tr_F P^{+k}$$

where the M_k are arbitrary. Each term in the sum forces $Tr_F P^k = 0$, and gives rise to a mass for the mode ϕ_k.

The confining behavior of Polyakov loop two-point functions at low temperatures is natural in the effective model.

Potentials of Type V_2

An important class of models is obtained from potential with pairwise interactions between the eigenvalues

$$V_2 = \sum_{j,k} v(\theta_j - \theta_k)$$

Two-loop perturbation theory gives this form (*Korthals Altes* 1994).

An elementary calculation shows

$$\sigma_n = \sqrt{\frac{2}{\kappa} \sum_{j=0}^{N-1} v^{(2)}\left(\frac{2\pi j}{N} \right) \sin^2\left(\frac{\pi n j}{N} \right)}$$

where $v^{(2)}$ is the second derivative of v. This is a master formula relating the string tensions to the underlying potential. (This is essentially the dispersion relation for a linear chain with arbitrary translation-invariant quadratic couplings: nearest-neighbot, next-nearest neighbor, *et cetera*.)

If the sum is dominated by the $j = 1$ term, representing a nearest-neighbor interaction in the space of eigenvalues, then we recover sine-law scaling

$$\sigma_n \simeq \sqrt{\frac{4}{\kappa} v^{(2)}\left(\frac{2\pi}{N} \right)} \sin\left(\frac{\pi n}{N} \right).$$

Casimir scaling can be obtained by a very small admixture of other components of $v^{(2)}$.

Sine-law scaling arises naturally from a nearest-neighbor interaction in the space of Polyakov loop eigenvalues.

Monte Carlo results for Casimir, Z(N), and sine-law scaling

We can check this by performing discrete Fourier transforms on ratios of string tensions. Data is normalized to give fractions of the total strength coming from nearest-neighbor, next-nearest-neighbor, *et cetera*, interactions.

	d	$\lvert j-k\rvert = 1$	$\lvert j-k\rvert = 2$	$\lvert j-k\rvert = 3$	$\lvert j-k\rvert = 4$	Ref
sine Law	any	1	0	0	0	
SU(4) Z(N)	any	0.941	0.059			
SU(4)	4	0.968(16)	0.032(7)			LTW
SU(4)	4	0.992(25)	0.008(5)			DPRV
SU(6) Z(N)	any	0.926	0.062	0.012		
SU(6)	4	0.960(16)	0.045(18)	-0.005(12)		LTW
SU(6)	4	0.996(40)	0.0003(218)	0.004(26)		DPRV
SU(8) Z(N)	any	0.925	0.058	0.013	0.004	
SU(8)	4	1.028(22)	-0.067(24)	0.047(32)	-0.009(22)	LTW
SU(4)	3	0.957(6)	0.043(2)			LT
SU(6)	3	0.930(5)	0.065(8)	0.004(5)		LT

LTW= *Lucini*, *Teper*, *and Wenger* (2004)
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Predictive Power of Scaling

In previous work on phenomenological models of the gluon equation of state (*Meisinger, Miller, and Ogilvie* 2002,2004) , we considered models of the form

$$f = -p = V(\theta) - 2\int \frac{d^3k}{(2\pi)^3} Tr_A \ln[1 - Pe^{-\beta\omega_k}]$$

where V is a phenomenologically chosen potential whose role is to favor confinement at low temperature. We studied two physically motivated potentials. One is a quadratic function of the eigenvalues

$$V_A = v_A \sum_{\alpha=2}^{N} \sum_{\beta=1}^{\alpha-1} (\theta_\alpha - \theta_\beta)(\theta_\alpha - \theta_\beta - 2\pi)$$

which appears as an $O(m^2T^2)$ term in the high temperature expansions at one-loop. This potential leads to $\sigma_n^A = \sigma_1$ for every N-ality. The other potential is the logarithm of Haar measure

$$V_B = v_B \sum_{\alpha=2}^{N} \sum_{\beta=1}^{\alpha-1} \ln[1 - \cos(\theta_\alpha - \theta_\beta)].$$

Such a potential might occur if Haar measure were responsible for confinement. This potential was first studied by *Dyson* (1962) in his fundamental work on random matrices. This potential leads to

$$\sigma_n^B = \sqrt{\frac{n(N-n)}{N-1}}\,\sigma_1$$

which one might call ”square root of Z(N) scaling”. These particular models are now ruled out at low temperature. It is interesting to note that there is a potential which gives $Z(N)$ scaling. It is the integrable potential

$$V_2(\theta) = \sum_{j\neq k} \frac{\lambda}{\sin^2\left(\frac{\theta_j - \theta_k}{2}\right)}$$

(*Calogero and Perelomov* (1978).

Information about the behavior of the different string tensions gives us information on the form of the potential V, **which in turn reflects the ultimate origin of confinement**.

$Z(N)$ Domain Walls at High T

In the deconfined phase, the $Z(N)$ symmetry breaks spontaneously, with N equivalent vacua characterized semiclassically by

$$Tr_F P = mz^p$$

Following *Bhattacharya et al*. (1992), consider the surface tension associated with a one-dimensional kink solution of the effective equations of motion which interpolates between different vacua. It suffices to consider solutions with the behavior

$$\lim_{x\to-\infty} Tr_F P = m$$

$$\lim_{x\to\infty} Tr_F P = mz^p$$

with $p = 1,..,\left[\frac{N}{2}\right]$. With each value of p, we can associate a surface tension ρ_k.

For high temperatures, we can consider the case where $m = N$, in which case the kink interpolates between different elements of the center of the group. There are two inequivalent straight-line paths within the Cartan algebra. The first has the form

$$P(x) = \exp\left[\frac{2\pi i}{N} Y_k q(x)\right]$$

where $Y_k = diag[k,..,k,N-k,..,N-k]$ with $N-k$ entries k and k entries $N-k$. The second is

$$P(x) = \exp\left[\frac{2\pi i}{N} kY_1 q(x)\right]$$

For all potentials of type V_2, the first path leads to

$$\rho_k = \frac{k(N-k)}{N-1}\rho_1$$

while the second leads to

$$\rho_k = k\rho_1$$

and is not favored. Similar behavior occurs in the Polyakov model in $d = 2+1$ dimensions for the string tension (*Kogan and Kovner* 2001). The effective potential, written is terms of dual variables, if of type V_2. It is natural to speculate that there is a class of self-dual effective models with identical spectral in both phases.

Summary

- There is a natural class of effective field theories for the Polyakov loop
- Confining behavior is naturally obtained
- For $SU(N)$, there are $N - 1$ string tensions associated with the $N - 1$ normal modes of the Polyakov loop eigenvalues
- Rich pattern of mixing of operators
- Sine-law scaling is a consequence of nearest-neighbor interactions between eigenvalues

Not Discussed

Mode mixing and the collapse of Casimir scaling
Representations with zero N-ality
Phase transitions
Large-N limit
Connection with underlying confinement mechanisms

Future Work

Lattice

Finer Operator Mixing Analysis
Variant models with $Z(N)$ symmetry: variant actions, Higgs
Direct reconstruction of V

Nonlattice

Corrections to sine-law scaling and $Z(N)$ scaling
Allowed phases of effective action
Duality